technical proof

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TP B.3
Throw Calibration and Contour Plots for Various Cut Angles, Speeds, English, and Roll

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
by David G. Alciatore, PhD, PE ("Dr. Dave")

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Created at the request of Colin Colenso

Equations from throw analysis (TP A.14):

$$R := \frac{1.125 \cdot in}{m}$$

$$v_{rel}(v, \omega_x, \omega_z, \phi) := \sqrt{\left(v \cdot \sin(\phi) - R \cdot \omega_z\right)^2 + \left(R \cdot \omega_x \cdot \cos(\phi)\right)^2}$$

$$\theta_{throw}(v, \omega_x, \omega_z, \phi, a, b, c) := \frac{\min\left[\frac{\left(a + b \cdot e^{-c \cdot v_{rel}(v, \omega_x, \omega_z, \phi)}\right) \cdot v \cdot \cos(\phi)}{v_{rel}(v, \omega_x, \omega_z, \phi)}, \frac{1}{7}\right] \cdot \left(v \cdot \sin(\phi) - R \cdot \omega_z\right)}{v \cdot \cos(\phi)} \cdot 36$$

$$\omega(v, PE) := \frac{5}{4} \cdot \frac{v}{R} \cdot PE$$

Definitions:

ω_x, ω_r: vertical plane roll rate (follow or draw)

ω_z, ω_e: English spin rate

PE_r: % roll (follow or draw)

PE_e: % English (side spin)

outside English (OE): ω_e, $PE_e > 0$

inside English (IE): ω_e, $PE_e < 0$

Speed scale:

soft:	$v_1 := 4 \frac{km}{hr}$	$v_1 = 1.111 \frac{m}{s}$	$v_1 = 2.485 \cdot mph$
medium-soft:	$v_2 := 8 \cdot \frac{km}{hr}$	$v_2 = 2.222 \frac{m}{s}$	$v_2 = 4.971 \cdot mph$
medium:	$v_3 := 12 \frac{km}{hr}$	$v_3 = 3.333 \frac{m}{s}$	$v_3 = 7.456 \cdot mph$
firm:	$v_4 := 20 \frac{km}{hr}$	$v_4 = 5.556 \frac{m}{s}$	$v_4 = 12.427 \cdot mph$
power:	$v_5 := 36 \frac{km}{hr}$	$v_5 = 10 \frac{m}{s}$	$v_5 = 22.369 \cdot mph$

Convert speeds to unitless m/s numbers:

$$v_1 := \frac{v_1}{\frac{m}{s}} \qquad v_2 := \frac{v_2}{\frac{m}{s}} \qquad v_3 := \frac{v_3}{\frac{m}{s}} \qquad v_4 := \frac{v_4}{\frac{m}{s}} \qquad v_5 := \frac{v_5}{\frac{m}{s}}$$

Calibration data (from experiment by Colin Colenso):

measured throw values (at speeds: 1, 2, 3):

cut angles: $\varphi_m := \begin{pmatrix} 30 \cdot deg \\ 45 \cdot deg \end{pmatrix}$

in inches per yard: $\theta_1 := \begin{pmatrix} 4.8 \\ 5.6 \end{pmatrix} \qquad \theta_2 := \begin{pmatrix} 3.5 \\ 3 \end{pmatrix} \qquad \theta_3 := \begin{pmatrix} 3 \\ 2.3 \end{pmatrix}$

in degrees: $\tan\left(\frac{\theta_1}{36}\right) = \begin{pmatrix} 7.685 \\ 8.985 \end{pmatrix} \cdot deg \qquad \tan\left(\frac{\theta_2}{36}\right) = \begin{pmatrix} 5.588 \\ 4.786 \end{pmatrix} \cdot deg \qquad \tan\left(\frac{\theta_3}{36}\right) = \begin{pmatrix} 4.786 \\ 3.666 \end{pmatrix} \cdot deg$

sum of squares of errors in calibration data:

$$\begin{aligned} SSE(a,b,c) := & \left(\theta_{throw}\left(v_1, 0, 0, \varphi_{m_0}, a, b, c\right) - \theta_{1_0}\right)^2 \ldots \\ & + \left(\theta_{throw}\left(v_2, 0, 0, \varphi_{m_0}, a, b, c\right) - \theta_{2_0}\right)^2 \ldots \\ & + \left(\theta_{throw}\left(v_3, 0, 0, \varphi_{m_0}, a, b, c\right) - \theta_{3_0}\right)^2 \ldots \\ & + \left(\theta_{throw}\left(v_1, 0, 0, \varphi_{m_1}, a, b, c\right) - \theta_{1_1}\right)^2 \ldots \\ & + \left(\theta_{throw}\left(v_2, 0, 0, \varphi_{m_1}, a, b, c\right) - \theta_{2_1}\right)^2 \ldots \\ & + \left(\theta_{throw}\left(v_3, 0, 0, \varphi_{m_1}, a, b, c\right) - \theta_{3_1}\right)^2 \end{aligned}$$

initial guesses for friction equation parameters: $a := 0.01 \quad b := 0.1 \quad c := 1$

find best fit of friction model to calibration data:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \mathrm{Minimize}(SSE, a, b, c) = \begin{pmatrix} 0.016 \\ 0.219 \\ 0.691 \end{pmatrix}$$

Throw equation in units of inches/yard:

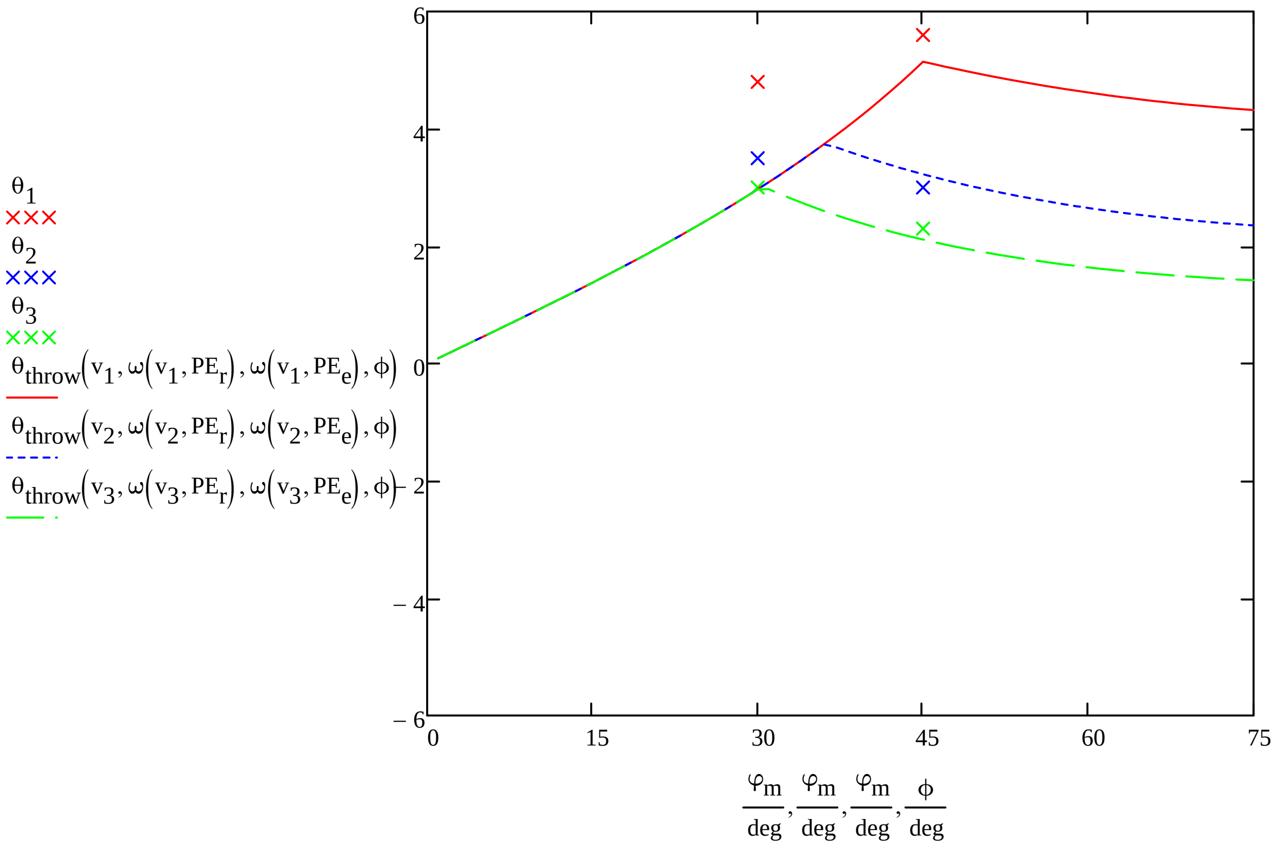
$$\theta_{throw}(v, \omega_x, \omega_z, \phi) := \frac{\min\left[\frac{\left(a + b \cdot e^{-c \cdot v_{rel}(v, \omega_x, \omega_z, \phi)}\right) \cdot v \cdot \cos(\phi)}{v_{rel}(v, \omega_x, \omega_z, \phi)}, \frac{1}{7}\right] \cdot \left(v \cdot \sin(\phi) - R \cdot \omega_z\right)}{v \cdot \cos(\phi)} \cdot 36$$

Plot of calibration data along with fit model:

$PE_r := 0\%$ $\qquad PE_e := 0\%$

$\theta_1 := \begin{pmatrix} 4.8 \\ 5.6 \end{pmatrix}$ $\qquad \theta_2 := \begin{pmatrix} 3.5 \\ 3 \end{pmatrix}$ $\qquad \theta_3 := \begin{pmatrix} 3 \\ 2.3 \end{pmatrix}$ $\qquad \varphi_m := \begin{pmatrix} 30 \cdot deg \\ 45 \cdot deg \end{pmatrix}$

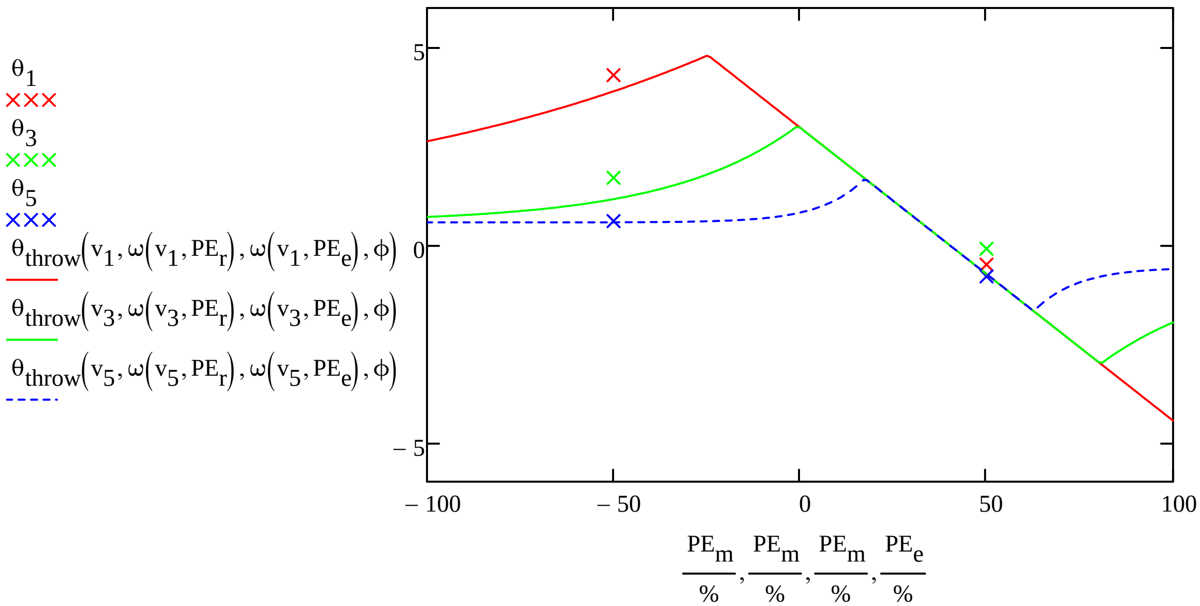
$\phi := 0 \cdot deg, 1 \cdot deg .. 75 \cdot deg$



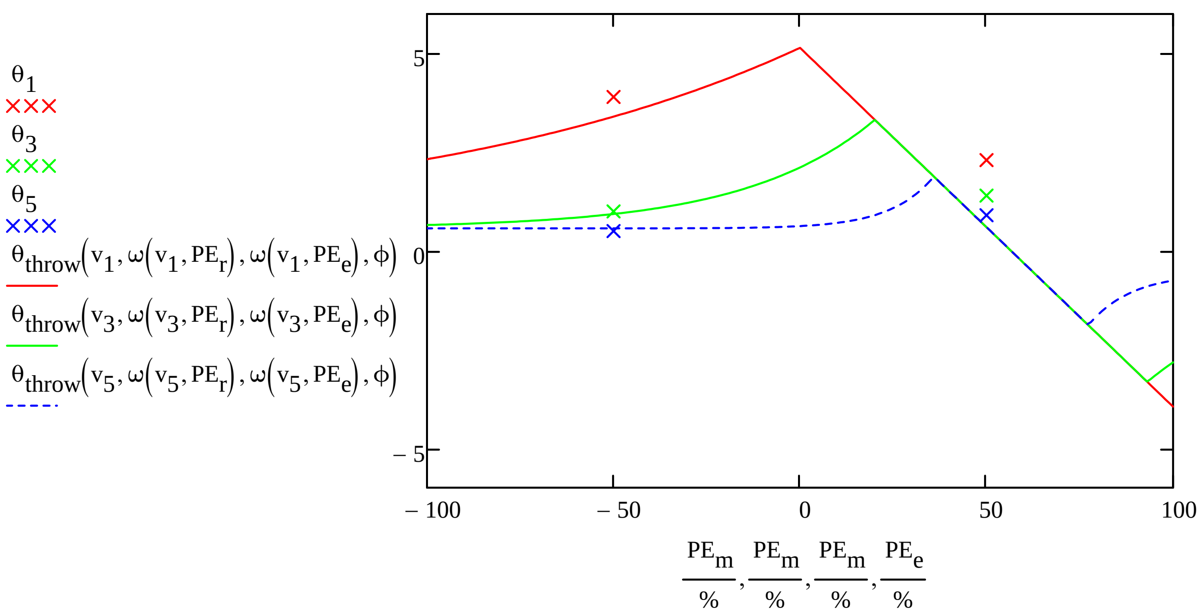
Plot other test data along with fit model:

$PE_r := 0\%$ $\qquad$ $PE_e := -100\cdot\%, -99\% .. 100\cdot\%$ $\qquad$ $PE_m := \begin{pmatrix} -50\% \\ 50\% \end{pmatrix}$

$\phi := 30\cdot deg$ $\qquad$ $\theta_1 := \begin{pmatrix} 4.3 \\ -.5 \end{pmatrix}$ $\qquad$ $\theta_3 := \begin{pmatrix} 1.7 \\ -.1 \end{pmatrix}$ $\qquad$ $\theta_5 := \begin{pmatrix} 0.6 \\ -.8 \end{pmatrix}$



$\phi := 45\cdot deg$ $\qquad$ $\theta_1 := \begin{pmatrix} 3.9 \\ 2.3 \end{pmatrix}$ $\qquad$ $\theta_3 := \begin{pmatrix} 1 \\ 1.4 \end{pmatrix}$ $\qquad$ $\theta_5 := \begin{pmatrix} .5 \\ .9 \end{pmatrix}$



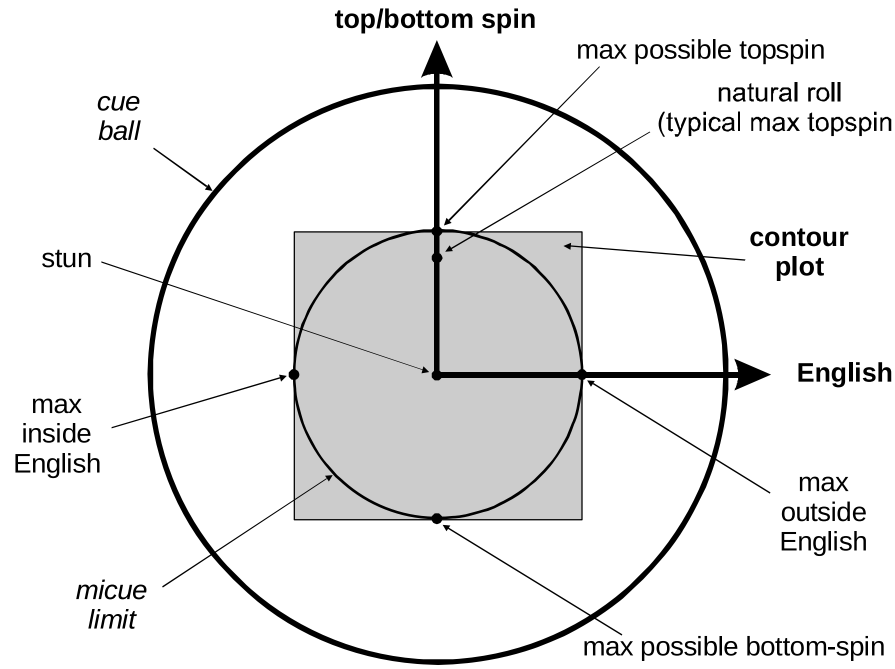
Generate contour plots of how throw varies with %English and %spin (top or bottom):

rectangular grid of %English and %spin:

$i := 1..41$ $\quad PE_{e_i} := -100\% + (i-1)\cdot 5\%$ $\quad PE_{e_1} = -100\cdot\%$ $\quad PE_{e_2} = -95\cdot\%$ $\quad PE_{e_{40}} = 95\cdot\%$ $\quad PE_{e_{41}} = 100\cdot\%$

$j := 1..21$ $\quad PE_{r_j} := 0\% + (j-1)\cdot 5\%$ $\quad PE_{r_1} = 0\cdot\%$ $\quad PE_{r_2} = 5\cdot\%$ $\quad PE_{r_{20}} = 95\cdot\%$ $\quad PE_{r_{21}} = 100\cdot\%$

NOTE - The figure below shows how all of the following contour plots should be interpreted:



<u>**contour plot color legend (ROYGBIV)**</u>**:**
red (max throw to right of line-of-centers)**, orange, yellow,**
green (very little or no throw)**,**
blue, indigo, violet (max throw to left of line-of-centers)

NOTE - The diagram above illustrates tip contact positions on the ball, but be aware that the throw values correspond to the spin on the CB at OB contact. Spin changes due to drag on the cloth, as the CB approaches the OB, must be taken into account. For example, the "maximum possible bottom-spin" point at the bottom is possible only for a firm shot over a short distance; otherwise, drag action reduces the bottom spin some, in which case the effective point on the diagram would be a little higher.

program to convert grid of calculated throw values to contour plot with fixed color range:

```
M_to_P(M, max_throw) := | for i ∈ 1..41
                        |   for j ∈ 1..21
                        |     | P[i-1, j+19] ← M[i, j]
                        |     | P[i-1, 21-j] ← M[i, j]
                        | P[0, 0] ← max_throw
                        | P[40, 40] ← -max_throw
                        | P
```

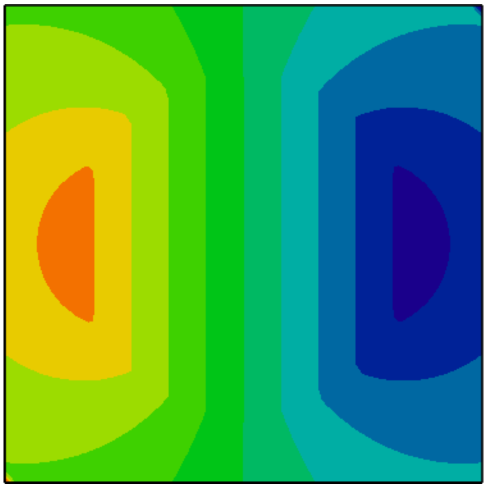
$max\_throw := 5.7$

full-ball hit (no cut):

$\varphi := 0.00001 \cdot deg$

$v := v_1 \qquad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



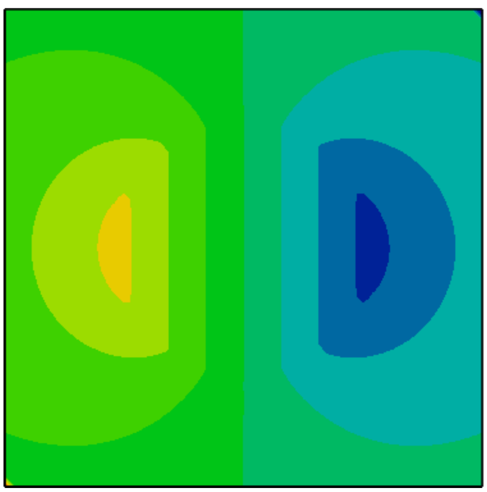
P

$max(M) = 4.5$

$min(M) = -4.5$

$v := v_2 \qquad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$

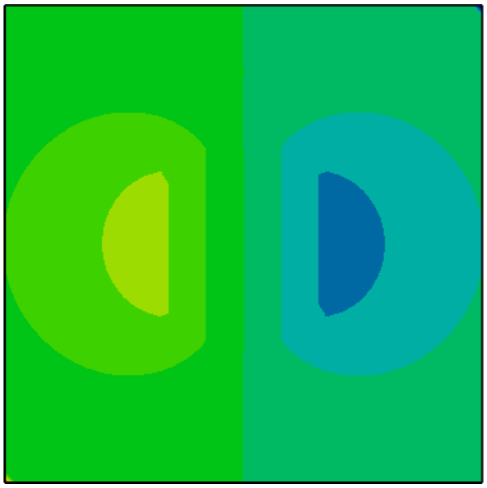


P

$max(M) = 3.308$

$min(M) = -3.308$

$v := v_3$ $\quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $\quad P := M\_to\_P(M, max\_throw)$

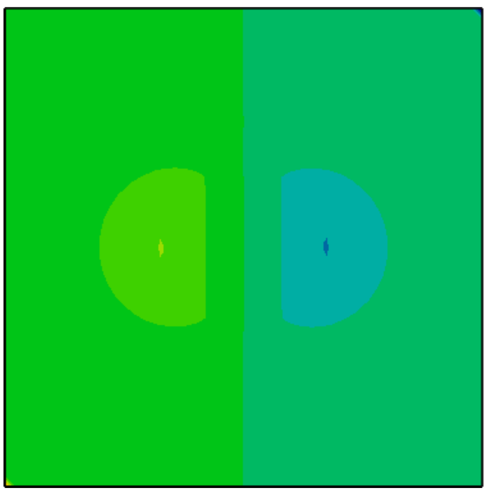


max(M) = 2.723

min(M) = −2.723

P

$v := v_4$ $\quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $\quad P := M\_to\_P(M, max\_throw)$

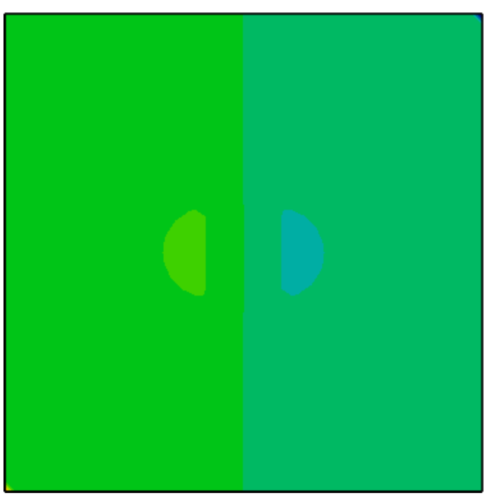


max(M) = 2.036

min(M) = −2.036

P

$v := v_5$ $\quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $\quad P := M\_to\_P(M, max\_throw)$



max(M) = 1.476

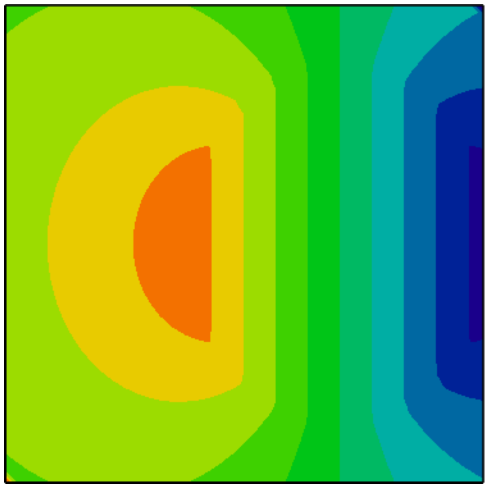
min(M) = −1.476

P

1/2-ball hit (medium cut):

$\varphi := 30 \cdot deg$

$v := v_1$ $\quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $\quad P := M\_to\_P(M, max\_throw)$

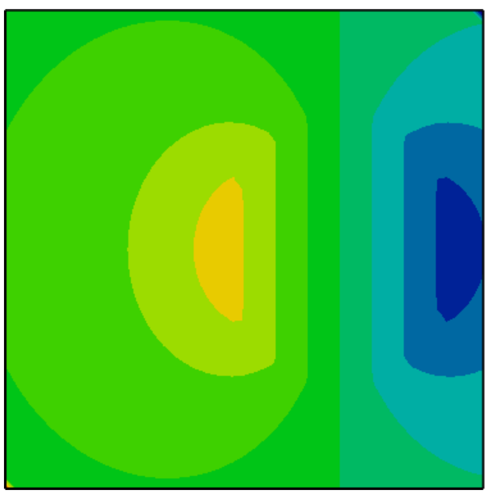


P

max(M) = 4.789

min(M) = −4.454

$v := v_2$ $\quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $\quad P := M\_to\_P(M, max\_throw)$



P

max(M) = 3.584

min(M) = −3.584

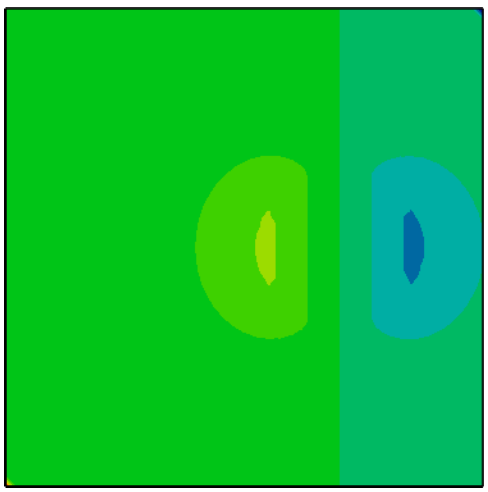
$v := v_3$ $M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $P := M\_to\_P(M, max\_throw)$

P

$max(M) = 2.969$

$min(M) = -2.969$

$v := v_4$ $M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $P := M\_to\_P(M, max\_throw)$

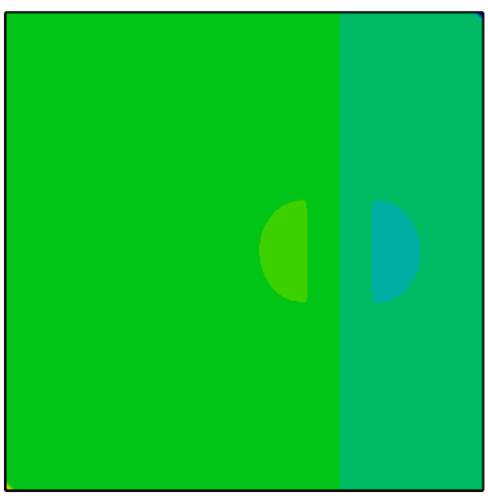


P

$max(M) = 2.227$

$min(M) = -2.227$

$v := v_5$ $M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$ $P := M\_to\_P(M, max\_throw)$



P

$max(M) = 1.485$

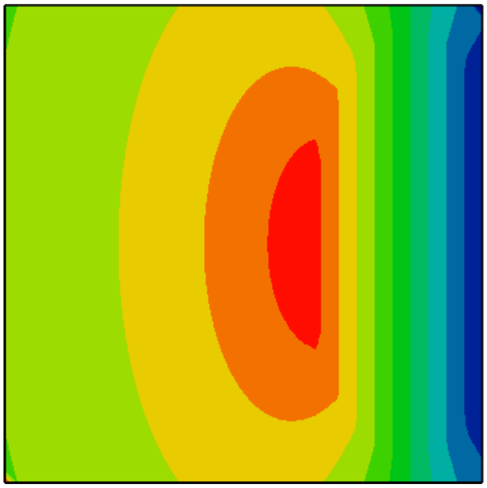
$min(M) = -1.485$

1/8-ball hit (thin cut):

$\varphi := 61.045 \cdot deg$

$v := v_1 \qquad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



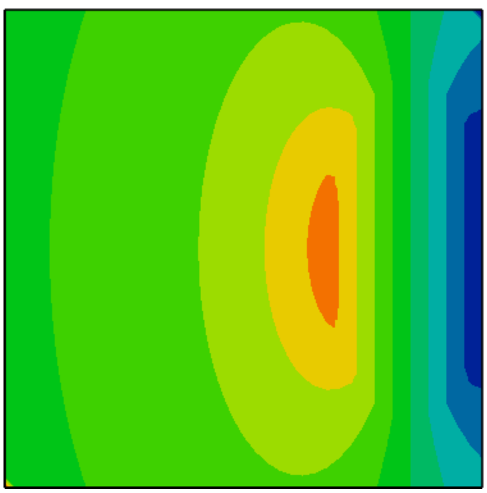
P

$max(M) = 5.682$

$min(M) = -3.984$

$v := v_2 \qquad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



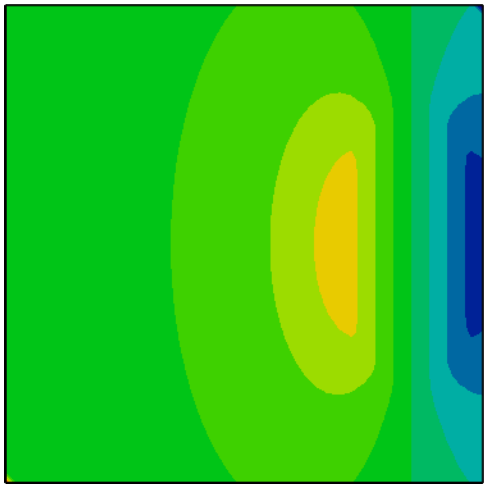
P

$max(M) = 4.591$

$min(M) = -3.984$

$\underset{\sim\sim}{v} := v_3 \quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



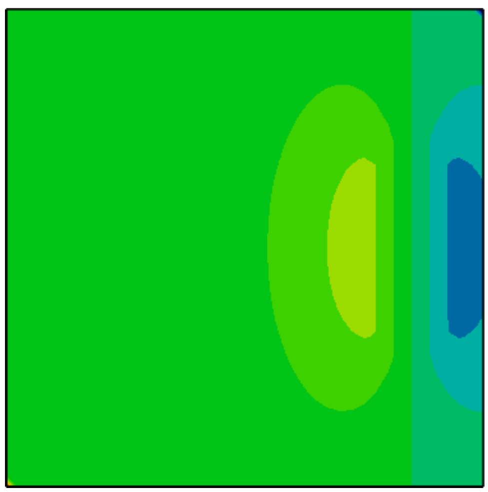
P

$max(M) = 3.888$

$min(M) = -3.888$

$\underset{\sim\sim}{v} := v_4 \quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



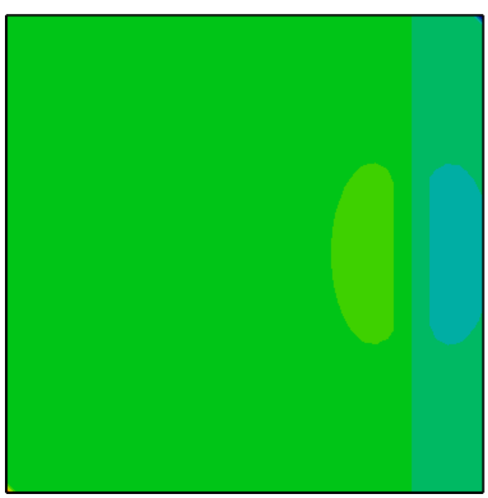
P

$max(M) = 2.94$

$min(M) = -2.94$

$\underset{\sim\sim}{v} := v_5 \quad M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$P := M\_to\_P(M, max\_throw)$



P

$max(M) = 1.992$

$min(M) = -1.992$

Write a dataset to a file for plotting outside of MathCAD:

$\varphi := 61.045 \cdot deg \qquad v := v_5$

$M_{i,j} := \theta_{throw}\left(v, \omega\left(v, PE_{r_j}\right), \omega\left(v, PE_{e_i}\right), \varphi\right)$

$M_{i,0} := \frac{PE_{e_i}}{\%} \qquad M_{0,j} := \frac{PE_{r_j}}{\%}$ row and column headings

WRITEPRN("data/eigth-5.txt") := M