TP A.8
The effects of English on the 30° rule

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu

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This technical proof is based on the background presented in TP A.6, assuming a perfect coefficient of restitution (e = 1). The Equation numbers below match the numbers in TP A.6. Only differences are shown, so not all Equations are included (see TP A.6 for details).

The total angular velocity of the cue ball, assuming it is rolling with English is:

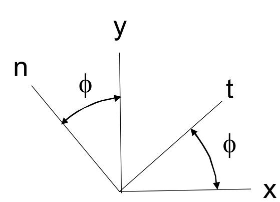
$$\vec{\omega} = -\frac{v}{R}\hat{i} + \omega\hat{k} \tag{1}$$

Therefore, the total velocity of the point of contact (B) between the cue ball and the object ball, at impact, is:

$$\vec{v}_B = \vec{v} + \vec{\omega}\times\vec{r}_{B/O} = v\hat{j} + (-\frac{v}{R}\hat{i} + \omega\hat{k})\times R(-\sin(\phi)\hat{i} + \cos(\phi)\hat{j}) = -R\omega\cos(\phi)\hat{i} + (v - R\omega\sin(\phi))\hat{j} - v\cos(\phi)\hat{k} \tag{3}$$

From the figure below, the t and n components of a velocity vector are related to the x and y components according to:

$$\vec{v} = v_t\hat{i} + v_n\hat{j} = (v_x\cos(\phi) + v_y\sin(\phi))\hat{t} + (v_y\cos(\phi) - v_x\sin(\phi))\hat{n}$$



Eliminating the normal component of the velocity in Equation 3, we can express the relative sliding velocity vector for the point of contact as:

$$\vec{v}_{rel} = [v\sin(\phi) - R\omega]\hat{t} - v\cos(\phi) \tag{4}$$

Therefore, the friction impulse can be expressed as:

$$\hat{F}_{fric} = -\mu_{balls}\hat{F}_n\frac{\vec{v}_{rel}}{|\vec{v}_{rel}|} = -\mu_{balls}mv\cos(\phi)\left\{\frac{[v\sin(\phi) - R\omega]\hat{t} - v\cos(\phi)\hat{k}}{\sqrt{[v\sin(\phi) - R\omega]^2 + (v\cos(\phi))^2}}\right\} \tag{6}$$

The term in the denominator of Equation 6 can be simplified to:

$$\sqrt{[v\sin(\phi)-R\omega]^2+(v\cos(\phi))^2}=\sqrt{v^2-2Rv\omega\sin(\phi)+R^2\omega^2}$$

Therefore, the tangential component of the friction impulse (reversing the sign to match the direction in TP A.5) is:

$$\hat{F}_{fric_t}=\mu_{balls}mv\cos(\phi)\left\{\frac{v\sin(\phi)-R\omega}{\sqrt{v^2-2Rv\omega\sin(\phi)+R^2\omega^2}}\right\} \tag{7}$$

and the vertical component is:

$$\hat{F}_{fric_z}=\mu_{balls}mv\cos(\phi)\left\{\frac{v\cos(\phi)}{\sqrt{v^2-2Rv\omega\sin(\phi)+R^2\omega^2}}\right\} \tag{8}$$

The tangential component of the post-impact cue ball velocity is:

$$v_{t0}=v\sin(\phi)-\frac{\hat{F}_{fric_t}}{m} \tag{9}$$

The post-impact angular speed about the x-axis is:

$$\omega_{xo}=\omega_x+\frac{R\cos(\phi)\hat{F}_{fric_z}}{\frac{2}{5}mR^2}=-\frac{v}{R}+\frac{5\mu_{balls}v^2\cos^3(\phi)}{2R\sqrt{v^2-2Rv\omega\sin(\phi)+R^2\omega^2}} \tag{16}$$

and about the y-axis is:

$$\omega_{yo}=\omega_y+\frac{R\sin(\phi)\hat{F}_{fric_z}}{\frac{2}{5}mR^2}=\frac{5\mu_{balls}v^2\sin(\phi)\cos^2(\phi)}{2R\sqrt{v^2-2Rv\omega\sin(\phi)+R^2\omega^2}} \tag{17}$$

The x and y components of the post-impact cue ball velocity can be expressed as:

$$v_{x0}=v_{t0}\cos(\phi)-v_{n0}\sin(\phi)=v\sin(\phi)\cos(\phi)(1-\mu_{balls}\cos(\phi)) \tag{18}$$

$$v_{y0}=v_{t0}\sin(\phi)+v_{n0}\cos(\phi)=v\sin^2(\phi)(1-\mu_{balls}\cos(\phi)) \tag{19}$$

Here are typical values for the parameters used in the equations along with the MathCAD forms of the results:

$\mu_{balls} := 0.06$ — average coefficient of friction between the balls

$\phi := 30 \cdot deg$ — half-ball hit

$\mu := 0.2$ — coefficient of friction between the cue ball and table cloth:

$g := \frac{g}{\left(\frac{m}{s^2}\right)}$ — acceleration due to gravity (in m/s^2) — $g = 9.807$

$v := 2$ — average pre-impact cue ball speed in m/s

$R := \frac{\frac{2.25}{2} \cdot in}{m}$ — ball radius (in meters)

$\omega := \frac{v}{R}$ — average English

initial cue ball velocity components:

$$\omega_{x0}(v,\omega,\phi,\mu_{balls}) := -\frac{v}{R} + \frac{5 \cdot \mu_{balls} \cdot v^2 \cdot \cos(\phi)^3}{2 \cdot R \cdot \sqrt{v^2 - 2 \cdot R \cdot v \cdot \omega \cdot \sin(\phi) + R^2 \cdot \omega^2}}$$

from Equation 16

$$\omega_{y0}(v,\omega,\phi,\mu_{balls}) := \frac{5 \cdot \mu_{balls} \cdot v^2 \cdot \sin(\phi) \cdot \cos(\phi)^2}{2 \cdot R \cdot \sqrt{v^2 - 2 \cdot R \cdot v \cdot \omega \cdot \sin(\phi) + R^2 \cdot \omega^2}}$$

from Equation 17

$$v_{x0}(v,\phi,\mu_{balls}) := v \cdot \sin(\phi) \cdot \cos(\phi) \cdot \left(1 - \mu_{balls} \cdot \cos(\phi)\right)$$

from Equation 18

$$v_{y0}(v,\phi,\mu_{balls}) := v \cdot \sin(\phi)^2 \cdot \left(1 - \mu_{balls} \cdot \cos(\phi)\right)$$

from Equation 19

v_C terms from TP A.4 used in several Equations:

$$v_{Cx0}(v,\omega,\phi,\mu_{balls}) := v_{x0}(v,\phi,\mu_{balls}) - R \cdot \omega_{y0}(v,\omega,\phi,\mu_{balls})$$

from Equation 22

$$v_{Cy0}(v,\omega,\phi,\mu_{balls}) := v_{y0}(v,\phi,\mu_{balls}) + R \cdot \omega_{x0}(v,\omega,\phi,\mu_{balls})$$

from Equation 23

$$v_{C0}(v,\omega,\phi,\mu_{balls}) := \sqrt{v_{Cx0}(v,\omega,\phi,\mu_{balls})^2 + v_{Cy0}(v,\omega,\phi,\mu_{balls})^2}$$

from Equation 24

time required for the cue ball to start rolling (cease sliding):

$$\Delta t(v, \omega, \phi, \mu_{balls}) := \frac{2 \cdot v_{C0}(v, \omega, \phi, \mu_{balls})}{7 \cdot \mu \cdot g}$$ from Equation 25

velocity components when the cue ball starts rolling in a straight line:

$$v_{xf}(v, \omega, \phi, \mu_{balls}) := \frac{1}{5} \cdot \left(5\, v_{x0}(v, \phi, \mu_{balls}) + 2 \cdot R \cdot \omega_{y0}(v, \omega, \phi, \mu_{balls})\right)$$ from Equation 26

$$v_{yf}(v, \omega, \phi, \mu_{balls}) := \frac{1}{5} \cdot \left(5 \cdot v_{y0}(v, \phi, \mu_{balls}) - 2 \cdot R \cdot \omega_{x0}(v, \omega, \phi, \mu_{balls})\right)$$ from Equation 27

the final deflected cue ball angle:

$$\theta_c(v, \omega, \phi, \mu_{balls}) := \operatorname{atan}\left(\frac{5 \cdot v_{x0}(v, \phi, \mu_{balls}) + 2 \cdot R \cdot \omega_{y0}(v, \omega, \phi, \mu_{balls})}{5 \cdot v_{y0}(v, \phi, \mu_{balls}) - 2 \cdot R \cdot \omega_{x0}(v, \omega, \phi, \mu_{balls})}\right)$$ from Equation 28

x position of the cue ball during the curved trajectory:

$$x_c(t, v, \omega, \phi, \mu_{balls}) := v_{x0}(v, \phi, \mu_{balls}) \cdot t - \frac{\mu \cdot g \cdot v_{Cx0}(v, \omega, \phi, \mu_{balls})}{2 \cdot v_{C0}(v, \omega, \phi, \mu_{balls})} \cdot t^2$$ from Equation 20

x position of the cue ball during and after the curve trajectory:

$$x(t, v, \omega, \phi, \mu_{balls}) := \left| \begin{array}{l} \Delta T \leftarrow \Delta t(v, \omega, \phi, \mu_{balls}) \\ x_c(t, v, \omega, \phi, \mu_{balls}) \ \text{ if } t \le \Delta T \\ \left[x_c(\Delta T, v, \omega, \phi, \mu_{balls}) + v_{xf}(v, \omega, \phi, \mu_{balls}) \cdot (t - \Delta T)\right] \ \text{ otherwise} \end{array} \right.$$

y position of the cue ball during the curved trajectory:

$$y_c(t, v, \omega, \phi, \mu_{balls}) := v_{y0}(v, \phi, \mu_{balls}) \cdot t - \frac{\mu \cdot g \cdot v_{Cy0}(v, \omega, \phi, \mu_{balls})}{2 \cdot v_{C0}(v, \omega, \phi, \mu_{balls})} \cdot t^2$$ from Equation 21

y position of the cue ball during and after the curve trajectory:

$$y(t, v, \omega, \phi, \mu_{balls}) := \left| \begin{array}{l} \Delta T \leftarrow \Delta t(v, \omega, \phi, \mu_{balls}) \\ y_c(t, v, \omega, \phi, \mu_{balls}) \ \text{ if } t \le \Delta T \\ \left[y_c(\Delta T, v, \omega, \phi, \mu_{balls}) + v_{yf}(v, \omega, \phi, \mu_{balls}) \cdot (t - \Delta T)\right] \ \text{ otherwise} \end{array} \right.$$

Parameters used in the plot below:

$T := 4$ number of seconds to display

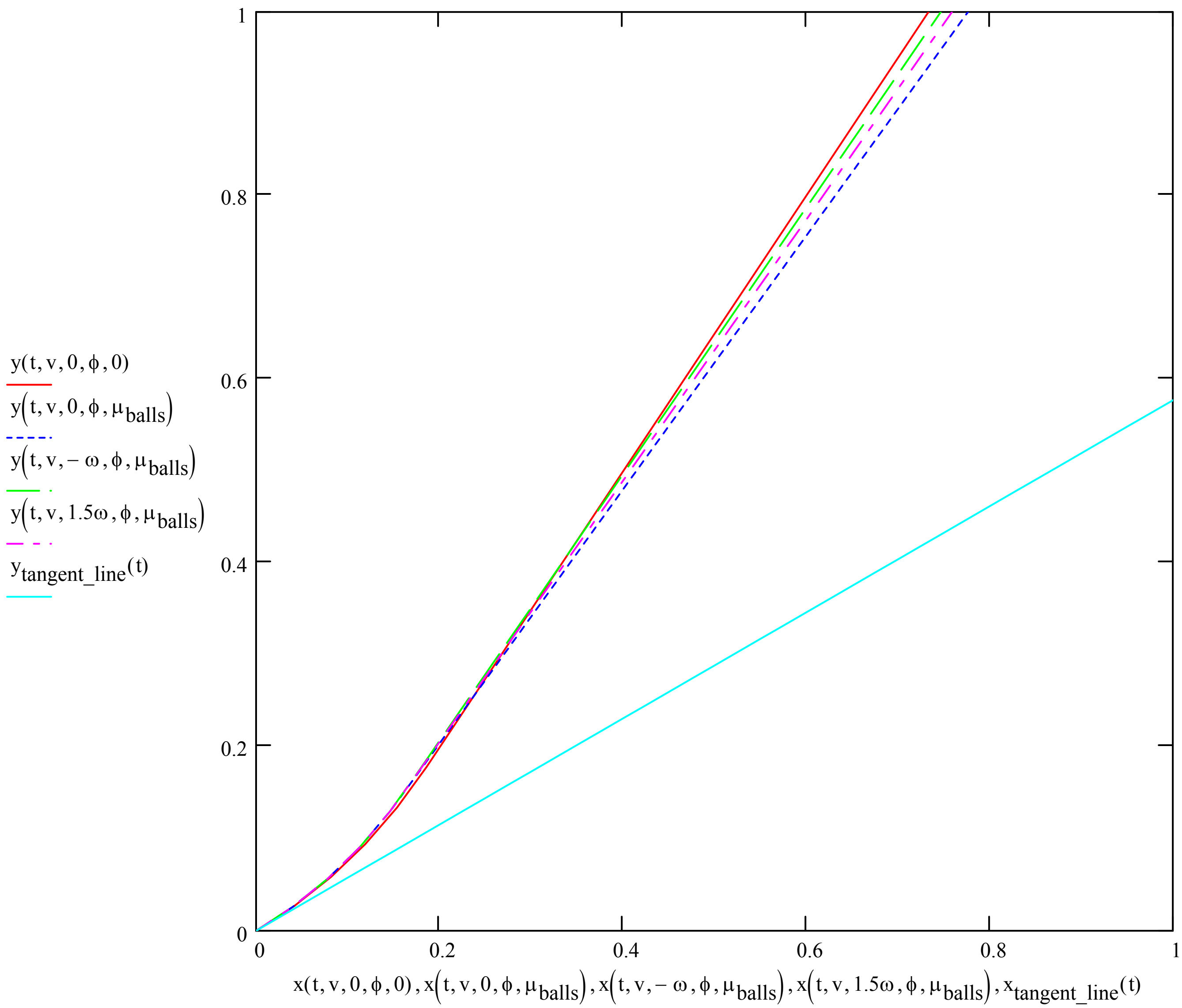
$t := 0, 0.05 .. T$ 0.1 second plotting increment

Equation for the tangent line:

$$x_{tangent\_line}(t) := \frac{t}{T} \cdot 2$$

$$y_{tangent\_line}(t) := x_{tangent\_line}(t) \cdot \tan(\phi)$$

half-ball hit with natural roll



final deflected cue ball angle:

$\theta_c(v, 0, \phi, 0) = 33.67\ \text{deg}$ with no friction or English

$\theta_c(v, 0, \phi, \mu_{balls}) = 35.907\ \text{deg}$ with no English (this agrees with TP A.6)

$\theta_c(v, \omega, \phi, \mu_{balls}) = 35.907\ \text{deg}$ with "gearing" outside English

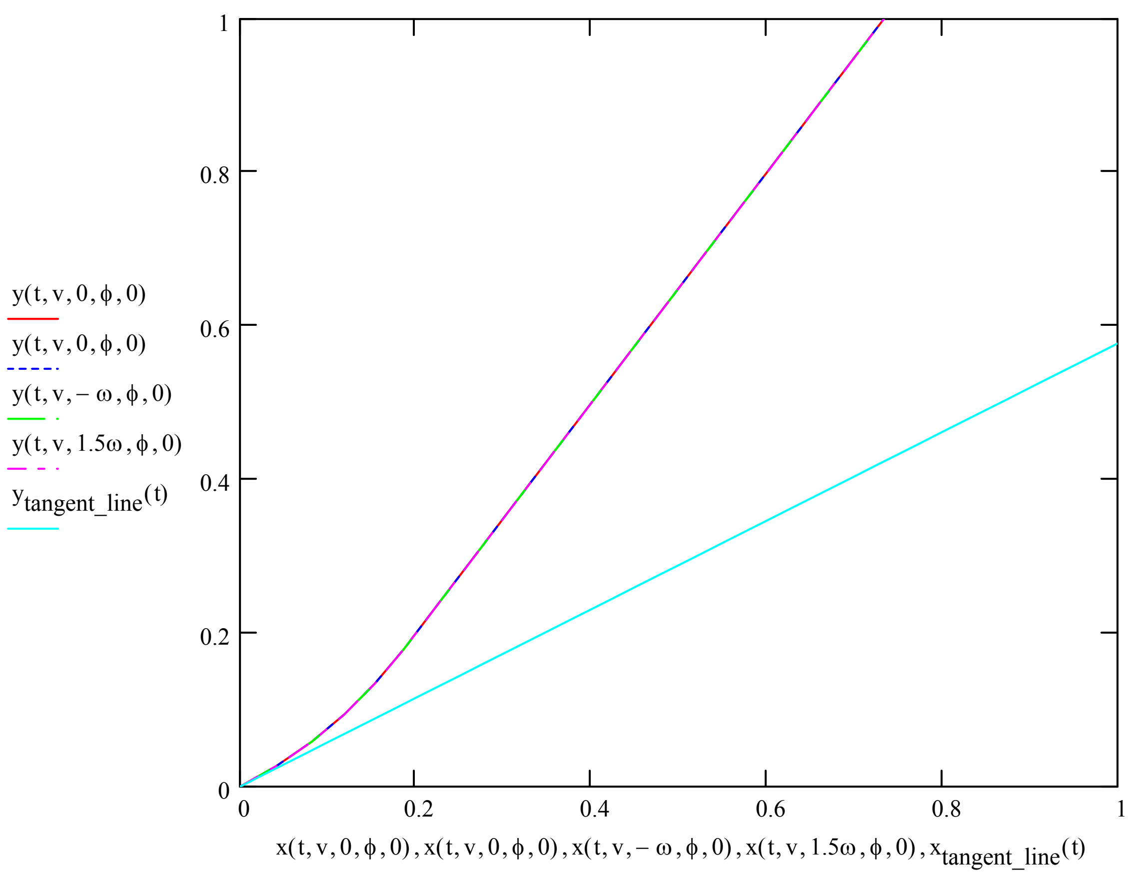
$\theta_c(v, 1.5\omega, \phi, \mu_{balls}) = 35.133\ \text{deg}$ with excess outside English

$\theta_c(v, -\omega, \phi, \mu_{balls}) = 34.574\ \text{deg}$ with inside English

$\theta_c(v, -2\omega, \phi, \mu_{balls}) = 33.955\ \text{deg}$ with excess inside English

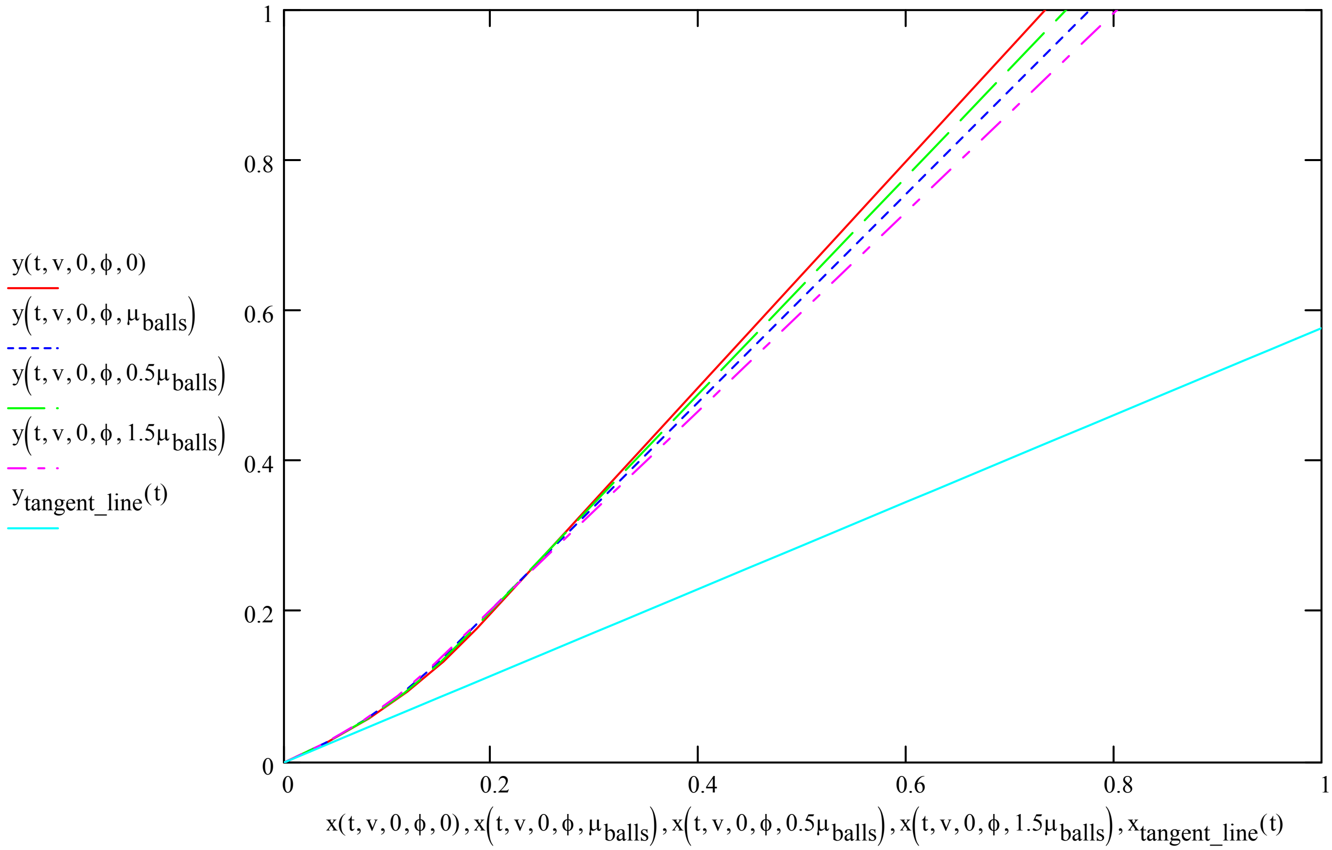
So excess outside English and inside English both shorten the cue ball angle, but not as much as the amount ball friction (without English) lengthens the angle. In any case, the effects are small.

The effects of English if there were no ball friction:



English would have no effect if there were no ball friction

The effects of changing ball friction on roll shots with no English:



The effects of changing ball friction on roll shots with inside English: