TP A.5
The effects of ball inelasticity and friction on the 90° rule

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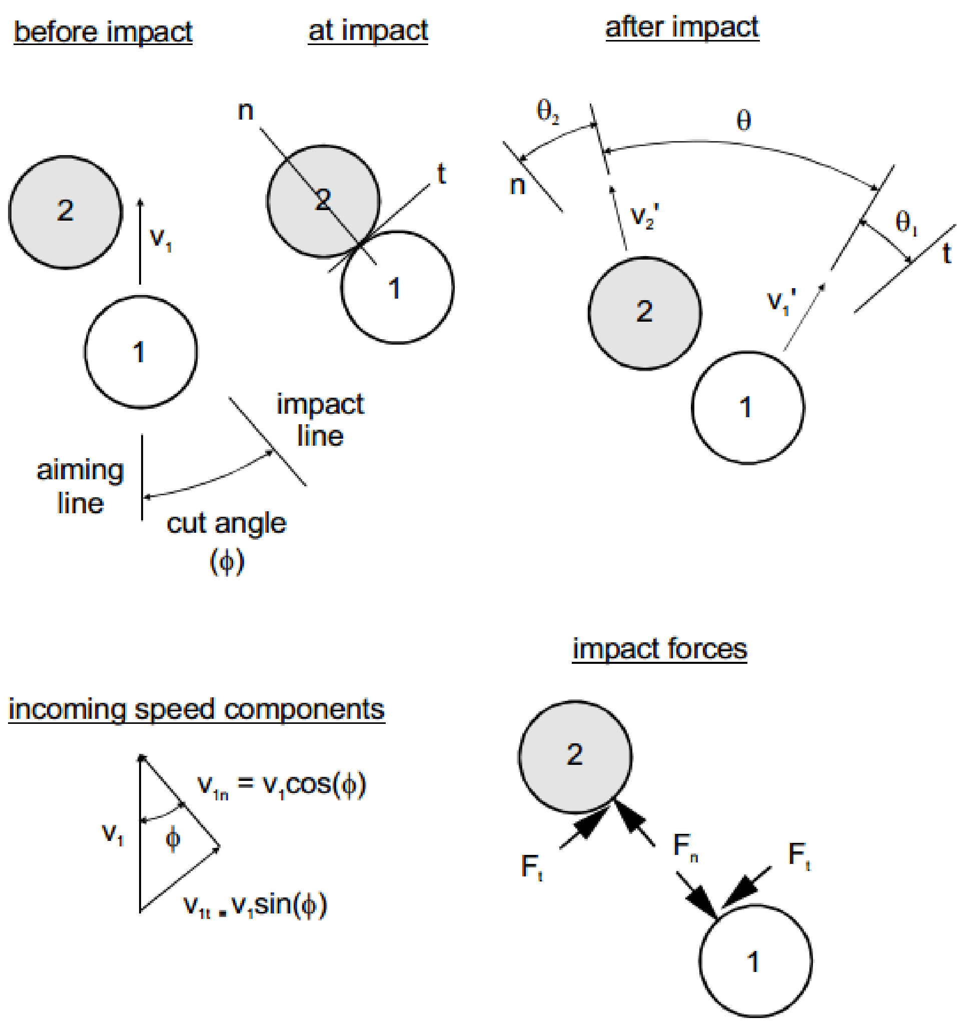
This technical proof looks at the effects of ball-to-ball coefficient of restitution (e) and friction (μ) on the 90 degree rule derived in TP 3.1.



NOTE: each ball is assumed to have equal mass (m)

Each ball has equal and opposite impulses in the normal (n) and tangential (t) directions given by:

$$\hat{F}_n = \int F_n dt \qquad and \qquad \hat{F}_t = \int F_t dt$$

The change in momentum of each ball in the n direction is equal to the normal impulse:

$$\hat{F}_n = m(v_{1n} - v_{1n}') = m(v_{2n}') \tag{1}$$

The change in momentum of each ball in the t direction is equal to the tangential impulse:

$$\hat{F}_t = m(v_{1t} - v_{1t}') = m(v_{2t}') \tag{2}$$

The speed of separation in the n direction is less than the speed of approach according to the coefficient of restitution (e):

$$v_{2n}' - v_{1n}' = e(v_{1n}) \tag{3}$$

The initial speed components, from the figure above, are:

$$v_{1t} = v_1 \sin(\phi) \tag{4}$$

$$v_{1n} = v_1 \cos(\phi) \tag{5}$$

where ϕ is the cut angle.

From Equation 1, we can write:

$$v_{2n}' = v_{1n} - v_{1n}' \tag{6}$$

Using this in Equation 3, with Equation 5, gives:

$$v_{1n}' = \frac{(1-e)}{2} v_{1n} = \frac{(1-e)\cos(\phi)}{2} v_1 \tag{7}$$

Substituting this back into Equation 1 gives:

$$\hat{F}_n = \frac{(1+e)}{2} m v_{1n} = \frac{(1+e)\cos(\phi)}{2} m v_1 \tag{8}$$

Using this in Equation 1, with Equation 5, gives:

$$v_{2n}' = \frac{(1+e)}{2} v_{1n} = \frac{(1+e)\cos(\phi)}{2} v_1 \tag{9}$$

Now we know both post-impact normal speed components (Equations 7 and 9).

The impulse in the t direction cannot reverse the direction of the relative tangential speed and it cannot exceed that allowed by friction, so from TP A.14 (Equation 15):

$$\hat{F}_t = \min\begin{pmatrix} \mu\hat{F}_n \\ mv_{1t} \end{pmatrix} = \min\begin{pmatrix} \left(\dfrac{\mu(1+e)\cos(\phi)}{2}\right) \\ \dfrac{1}{7}\sin(\phi) \end{pmatrix} mv_1 \quad (10)$$

I want to thank Sorokin Alexander for pointing out an error in Equation 10 in the original version of this document. The equation is now more accurate; although, the results below are unchanged.

Using this in Equation 2, we can solve for the post-impact tangential speeds:

$$v_{2t}' = \min\begin{pmatrix} \left(\dfrac{\mu(1+e)\cos(\phi)}{2}\right) \\ \dfrac{1}{7}\sin(\phi) \end{pmatrix} v_1 \quad (11)$$

$$v_{1t}' = v_{1t} - v_{2t}' = \left(\sin(\phi) - \min\begin{pmatrix} \left(\dfrac{\mu(1+e)\cos(\phi)}{2}\right) \\ \dfrac{1}{7}\sin(\phi) \end{pmatrix} \right) v_1 \quad (12)$$

So now we can determine the direction of each ball after impact, along with the angle between their paths (see the "after impact" figure above):

$$\theta_1 = \tan^{-1}\left(\frac{v_{1n}'}{v_{1t}'}\right) = \tan^{-1}\left(\frac{(1-e)\cos(\phi)}{2\sin(\phi) - \min\begin{pmatrix} \mu(1+e)\cos(\phi) \\ \dfrac{2}{7}\sin(\phi) \end{pmatrix}} \right) \quad (13)$$

$$\theta_2 = \tan^{-1}\left(\frac{v_{2t}'}{v_{2n}'}\right) = \tan^{-1}\left(\frac{\min\begin{pmatrix} \mu(1+e)\cos(\phi) \\ \dfrac{2}{7}\sin(\phi) \end{pmatrix}}{(1+e)\cos(\phi)} \right) \quad (14)$$

$$\theta = 90^\circ - \theta_1 - \theta_2 \quad (15)$$

Now putting the equations in MathCAD form and entering typical data:

$e := 0.94$ coefficient of restitution between balls

$\mu := 0.06$ average coefficient of friction between balls

$\phi := 30 \cdot deg$ half-ball hit

$$\theta_1(e,\mu,\phi) := \text{angle}\left(\left(2 \cdot \sin(\phi) - min\left(\begin{bmatrix} \mu \cdot (1+e) \cdot \cos(\phi) \\ \frac{2}{7} \cdot \sin(\phi) \end{bmatrix}\right)\right), (1-e) \cdot \cos(\phi)\right)$$ (from Equation 13)

$$\theta_2(e,\mu,\phi) := \text{angle}\left((1+e) \cdot \cos(\phi), min\left(\begin{bmatrix} \mu \cdot (1+e) \cdot \cos(\phi) \\ \frac{2}{7} \cdot \sin(\phi) \end{bmatrix}\right)\right)$$ (from Equation 14)

$$\theta(e,\mu,\phi) := 90 \cdot deg - \theta_1(e,\mu,\phi) - \theta_2(e,\mu,\phi)$$ (from Equation 15)

With no inelasticity or friction:

$\theta_1(1,0,\phi) = 0\ deg$ $\theta_2(1,0,\phi) = 0\ deg$ $\theta(1,0,\phi) = 90\ deg$

This is the 90 degree rule result presented in TP 3.1.

With inelasticity only:

$\theta_1(e,0,\phi) = 2.975\ deg$ $\theta_2(e,0,\phi) = 0\ deg$ $\theta(e,0,\phi) = 87.025\ deg$

inelasticity "shortens" the cue ball angle

With friction only:

$\theta_1(1,\mu,\phi) = 0\ deg$ $\theta_2(1,\mu,\phi) = 3.434\ deg$ $\theta(1,\mu,\phi) = 86.566\ deg$

friction "shortens" the object ball angle (this is called "throw")

With inelasticity and friction:

$\theta_1(e,\mu,\phi) = 3.307\ deg$ $\theta_2(e,\mu,\phi) = 3.434\ deg$ $\theta(e,\mu,\phi) = 83.259\ deg$

So the 90 degree rule is actually something less than the 90 degree rule.
e and μ vary with shot speed and cut angle in practice; but in all cases,
the actual angle between the ball paths will be less than 90 degrees.